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ON A CLASS OF INTERNAL SOLITARY WAVES IN A TWO-LAYER FLUID

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Solitary waves on an interface between two fluids are considered. A uniform asymptotic expansion is constructed for internal solitary waves with flat crests (of the plateau type) that degenerate into a bore in the limit. It is shown that, in this case, in contrast to a Korteweg-de Vries wave, the wave amplitude is of the same order of smallness as the longwave approximation parameter.

Ovsyannikov [1, Chap. 1] classified possible types of stationary waves in a two-layer fluid under a rigid lid within the second approximation of shallow water theory, and Funakoshi [2] gave a classification of these waves using the Korteweg–de Vries equation with quadratic and cubic nonlinearities. The existence theorem for solitary waves in a two-layer fluid (in an exact formulation) was proved by Khabakhpasheva [3] and Amick and Turner [4]. The existence of bore-like solutions of Euler equations was proved by Amick and Turner [5] and Makarenko [6].

Funakoshi and Oikawa [7] and Turner and Vanden-Broeck [8] studied numerically solitary waves in two-layer flows without shear that become a bore as their amplitude and velocity tend to critical values. Mirie and Pennell [9] analyzed this situation by semianalytical methods for a long-wave approximation of ninth-order accuracy in amplitude.

1. Basic Equations. Steady two-dimensional irrotational flow of an ideal incompressible two-layer fluid in a gravity field is considered. It is assumed that at infinity the velocities of the layers are U_i, where $i = 1$ and 2 (subscript 1 refers to the lower layer and subscript 2 refers to the upper layer). The flow domain is a strip of width $H = H_1 + H_2$ divided by a contact discontinuity line γ_1 into two curvilinear strips Ω_i. The fluid is bounded by a flat bottom (γ_0: $y = 0$) and a flat rigid lid (γ_2: $y = H$).

The equations of motion written in terms of the stream function are

$$\begin{aligned} &\psi_{xx} + \psi_{yy} = 0 && \text{in} \quad \Omega_1 \cup \Omega_2, \\ &\psi = 0 && \text{on} \quad \gamma_0, \\ &\psi = Q_1, \quad [\rho(\psi_x^2 + \psi_y^2 + 2gy - 2b)] = 0 && \text{on} \quad \gamma_1, \\ &\psi = Q_1 + Q_2 && \text{on} \quad \gamma_2, \\ &\psi \to \psi_0, \quad \nabla\psi \to \nabla\psi_0, && |x| \to \infty. \end{aligned} \tag{1}$$

Here brackets denote a jump of a corresponding quantity at the interface between the layers, $Q_i = U_i H_i$ and ρ_i are the discharge rates and densities in the layers, b is the Bernoulli constant, and ψ_0 is the stream function of the unperturbed piecewise-constant flow:

$$\psi_0 = \begin{cases} yQ_1/H_1, & 0 < y < H_1, \\ Q_1 + (y - H_1)Q_2/H_2, & H_1 < y < H. \end{cases}$$

The problem is to find a nontrivial solution different from this flow.

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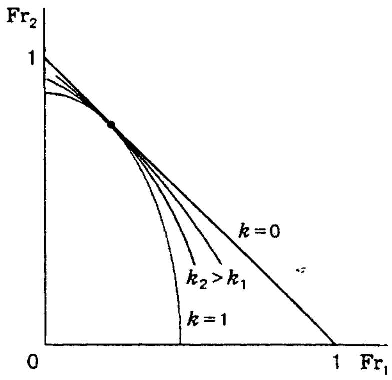

Fig. 1

We introduce dimensionless variables using H_1 as the length scale, Q_1 as the scale for the stream function in the lower layer, and Q_2/r ($r = H_2/H_1$) as the scale for the stream function in the upper layer. The characteristic dimensionless parameters of the problem are the Froude numbers

$$\mathrm{Fr}_i^2 = \frac{U_i^2}{gH}\frac{\rho_i}{\rho_1-\rho_2}.$$

The Mises transformation $(x, y) \to (x, \psi)$ maps the flow domain onto the double strip

$$\Pi = \Pi_1 \cup \Pi_2, \qquad \Pi_1 = (0,1)\times\mathbb{R}, \qquad \Pi_2 = (1, 1+r)\times\mathbb{R},$$

where r is the ratio of the unperturbed depths of the layers.

We seek streamlines of the form

$$y = \psi + \varepsilon w(\tilde{x}, \psi), \qquad \tilde{x} = \varepsilon x,$$

where ε is a long-wave approximation parameter, which is defined below.

In the dimensionless variables, the system of equations for w takes the form

$$\begin{aligned} &\varepsilon^2 w_{xx} + w_{\psi\psi} = \operatorname{div} \boldsymbol{Q}(\nabla w, \varepsilon), && 0<\psi<1, \quad 1<\psi<1+r,\\ &w = 0, && \psi = 0, \quad \psi = h,\\ &[w] = 0, \quad [\mathrm{Fr}^2 w_\psi] + h^{-1} w = [\mathrm{Fr}^2 Q_2(\nabla w)], && \psi = 1,\\ &w \to 0, \quad \nabla w \to 0, && |x| \to \infty, \end{aligned} \tag{2}$$

where

$$\boldsymbol{Q} = (Q_1, Q_2) = \left(\frac{\varepsilon^3 w_x w_\psi}{1+\varepsilon w_\psi}, \frac{1}{2}\frac{\varepsilon^3 w_x^2 + 3\varepsilon w_\psi^2 + 2\varepsilon^2 w_\psi^3}{(1+\varepsilon w_\psi)^2}\right),$$

and $\mathrm{Fr} = \mathrm{Fr}_i$ in Π_i.

2. Preliminary Analysis. The problem (2) linearized over the piecewise-constant flow with specified Fr_i and r have solutions in the form of elementary wave packets

$$w(x, \psi) = W(\psi)\, e^{-i æ x},$$

if æ is related to the parameters of the main flow by the dispersion relation

$$\Delta(æ) \equiv \mathrm{Fr}_1^2 æ \coth æ + \mathrm{Fr}_2^2 æ \coth(ræ) - (r+1)^{-1} = 0. \tag{3}$$

Real roots of Eq. (3) exist only if

$$\mathrm{Fr}_1^2 + r^{-1}\mathrm{Fr}_2^2 \leqslant (1+r)^{-1}. \tag{4}$$

This inequality defines a spectrum of linear waves. The shaded region in Fig. 1 corresponds to inequality (4). The boundary of the spectrum consists of bifurcation points for solutions of the problem. At these points branching of long stationary waves from the piecewise-constant flow occurs. In particular, it is shown [1, 6] that for

$$|\mathrm{Fr}_1| + |\mathrm{Fr}_2| = 1 \tag{5}$$

there exists a one-parameter solution of the problem in the form of a bore. We shall consider the family of solutions for pairs of Froude numbers (Fr_1 and Fr_2) lying in the supercritical region (outside the ellipse) in the neighborhood of the point

$$(\overset{0}{F}_1, \overset{0}{F}_2) = (1/(1+r), r/(1+r)). \tag{6}$$

We introduce the parametrization $(\mathrm{Fr}_1, \mathrm{Fr}_2) \mapsto (\varepsilon, k)$ by the following rule. Let ε be the least positive root of the equation

$$\mathrm{Fr}_1^2 \varepsilon \cot \varepsilon + \mathrm{Fr}_2^2 \varepsilon \cot (r\varepsilon) - (r+1)^{-1} = 0. \tag{7}$$

For the points $(\mathrm{Fr}_1, \mathrm{Fr}_2)$ outside the ellipse (4), the dispersion function $\Delta(æ)$ has only purely imaginary conjugate roots, and $æ = \pm i\varepsilon$ are the roots nearest to the real axis. The parameter ε is the index of exponential decay of the solution at infinity. Equation (7) is an analog of Stokes's formula for solitary surface waves $\mathrm{Fr}^2 = \tan\varepsilon/\varepsilon$.

The family of level curves of the second parameter k consists of the ellipses

$$\left(k\mathrm{Fr}_1 + \frac{1-k}{r+1}\right)^2 + \frac{1}{r}\left(k\mathrm{Fr}_2 + \frac{r(1-k)}{r+1}\right)^2 - \frac{1}{r+1} = 0, \tag{8}$$

each of which is tangent to the sides of the square (5) at the point (6). As k varies from 0 to 1, these ellipses occupy the curvilinear sectors between the straight line $\mathrm{Fr}_1 + \mathrm{Fr}_2 = 1$ and the ellipse (4) (see Fig. 1).

The parametrization (7), (8) has a singularity at the point $(\overset{0}{F}_1, \overset{0}{F}_2)$, which is a consequence of the behavior of the solution in the neighborhood of the point. The Froude numbers are expanded in power series in ε

$$\mathrm{Fr}_i = \sum_{n=0}^{\infty} \varepsilon^n \overset{n}{F}_i,$$

where the coefficients of the series are given by

$$\overset{1}{F}_1 = \frac{1}{\theta(1+r)\sqrt{1-k}}, \qquad \overset{1}{F}_2 = -\overset{1}{F}_1,$$
$$\overset{2}{F}_1 = \frac{1}{6}\,\frac{2r^2 + kr^2 - 2r - kr + k}{(k-1)(r+1)}, \qquad \overset{2}{F}_2 = \frac{1}{6}\,\frac{r(r^2k - 2r - kr + k + 2)}{(k-1)(r+1)}. \tag{9}$$

Here $\theta = \sqrt{3(r+1)/(r(r^3+1))}$.

3. Coefficients of the Perturbation Series. Substitution of w in the form of the power series in ε

$$w = \sum_{n=0}^{\infty} \varepsilon^n w_n$$

into (2) yields the following sequence of systems of equations for w_n:

$$w_{n\psi\psi} = g_n \quad \text{in} \quad \Pi_i; \tag{10}$$

$$w_0 = 0 \quad \text{for} \quad \psi = 0, h; \tag{11}$$

$$[w_n] = 0 \quad \text{for} \quad \psi = 1; \tag{12}$$

$$\Lambda w_n = \varphi_n \quad \text{for} \quad \psi = 1. \tag{13}$$

Here the differential operator Λ has the form

$$\Lambda w = \overset{0}{F_2^2}\, w_\psi(x, 1+0) - \overset{0}{F_1^2}\, w_\psi(x, 1-0) + h^{-1} w(x, 1),$$

the right sides of (10) are

$$g_0 = 0, \qquad g_1 = \frac{3}{2}(w_{0\psi}^2)_\psi, \qquad g_2 = -w_{0xx} + (3w_{0\psi}w_{1\psi} - 2w_{0\psi})_\psi,$$

$$g_n = -w_{n-2,xx} + [3(w_{0\psi}w_{n-1,\psi} + w_{1\psi}w_{n-2,\psi}) - 8w_{0\psi}^2 w_{n-2,\psi}]_\psi + f_1(w_0, \ldots, w_{n-3}),$$

and the right sides of (13) are given by

$$\varphi_0 = 0, \qquad \varphi_1 = -[2\, \overset{0}{F}\overset{1}{F}\, w_{0\psi}] + \left[\frac{3}{2} w_{0\psi}^2\, \overset{0}{F^2}\right],$$

$$\varphi_2 = -[\overset{1}{F^2}\, w_{0\psi} + 2\, \overset{0}{F}\overset{1}{F}\, w_{1\psi} + 2\, \overset{0}{F}\overset{2}{F}\, w_{0\psi}] + [3\, \overset{0}{F^2}\, w_{0\psi}w_{1\psi} - 2\, \overset{0}{F^2}\, w_{0\psi}^3 + 3\, \overset{0}{F}\overset{1}{F}\, w_{0\psi}^2],$$

$$\varphi_n = [6\, \overset{0}{F}\overset{1}{F}\, w_{0\psi}w_{n-2,\psi} + 3\, \overset{0}{F^2}\, (w_{0\psi}w_{n-1,\psi} + w_{1\psi}w_{n-2,\psi}) - 6\, \overset{0}{F^2}\, w_{0\psi}^2 w_{n-2,\psi}]$$

$$- [2\, \overset{0}{F}\overset{1}{F}\, w_{n-1,\psi} + (2\, \overset{0}{F}\overset{2}{F} + \overset{1}{F^2}) w_{n-2,\psi}] + f_2(w_0, \ldots, w_{n-3}).$$

The solutions of Eqs. (10)-(13) for $n = 1$ and $n = 2$ are

$$w_0(x, \psi) = C_0(x)W(\psi), \qquad w_1(x, \psi) = C_1(x)W(\psi),$$

where

$$W = \begin{cases} \psi, & 0 < \psi < 1, \\ (\psi - h)/(1 - h), & 1 < \psi < h, \end{cases}$$

$h = 1 + r$, and the functions $C_0(x)$ and $C_1(x)$ remain undetermined and are obtained from the compatibility condition for equations of higher-order approximations.

The solutions of Eqs. (10)-(13) for $n \geqslant 2$ have the structure

$$w_n(x, \psi) = C_n(x)W(\psi) + C_{n-2}(x)W_1(\psi) + f(C_0, \ldots, C_{n-3}),$$

where

$$W_1 = \begin{cases} \psi^3/6, & 0 < \psi < 1, \\ (\psi - h)^3/(6r^2), & 1 < \psi < h. \end{cases}$$

The following equation for the function $C_0(x)$ is obtained from (13) for $n = 2$:

$$C_0'' = P_3(C_0), \qquad P_3(C_0) = \frac{6h}{r(r^3+1)} C_0^3 - \frac{9h^2}{r(r^3+1)} \overset{1}{F_1} C_0^2 + C_0. \tag{14}$$

Hence,

$$C_0 = a \frac{1 - \tanh^2(x/2)}{\theta(a^2 - \tanh^2(x/2))},$$

where $a + 1/a = \pm 2/\sqrt{1-k} = \pm 2(1+r)\overset{1}{F_1}\theta$. The plus sign corresponds to a wave level rise wave, and the minus sign corresponds to a depression wave.

The functions C_n for higher-order approximations are obtained from the recurrence formula

$$C_n'' - P_3'(C_0)C_n = f_n(C_0, \ldots, C_{n-1}). \tag{15}$$

For a flow without shear, it is shown [10] that any solution of the original problem of internal waves with symmetric conditions at infinity is symmetric about the vertical axis. The even solution of Eq. (15) is uniquely

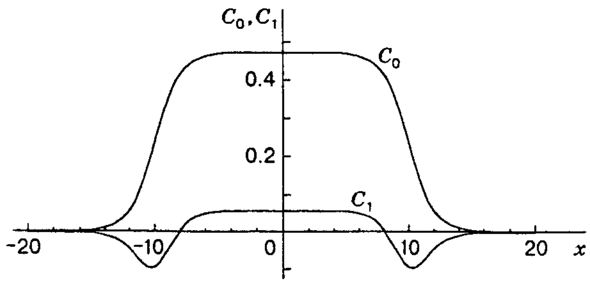

Fig. 2

determined and given by

$$C_n = -u_1 \int\limits_0^x f_n u_2 \, dx - u_2 \int\limits_x^{+\infty} f_n u_1 \, dx.$$

Here $u_{1,2}$ are fundamental solutions of the homogeneous equation, $u_1 = C_0'$, and u_2 is expressed in terms of u_1 using the Liouville formula and has the form

$$u_2 = \frac{\beta_1 \cosh 3x + \beta_2 \cosh 2x + \beta_3 x \sinh x + \beta_4}{(\beta_5 + \beta_6 \cosh x)^2},$$

where $\beta_1 = (a^6 + 3a^4 + 3a^2 - 1)(a^2 - 1)$, $\beta_2 = 16(a^2 - 1)^3(a^2 + 1) - 65a^8 + 4a^6 - 6a^4 + 4a^2 - 65$, $\beta_3 = (60a^6 + 12a^4 - 12a^2 - 60)(a^2 - 1)$, $\beta_4 = (-80a^6 - 48a^4 - 48a^2 + 80)(a^2 - 1)$, $\beta_5 = \sqrt{16a(a^2 - 1)/\theta}(a^2 + 1)$, and $\beta_6 = \sqrt{16a(a^2 - 1)/\theta}(a^2 - 1)$.

For $n = 1$, we have $f_1 = k_4 C_0^4 + k_3 C_0^3 + k_2 C_0^2$ with the coefficients given by

$$k_2 = \frac{(1 - r)(r^2 - kr + 2r + 1)}{2\theta^2 r^2 (r + 1)^2 (k - 1)}, \quad k_3 = \frac{4}{\theta^3 r(r + 1)\sqrt{1 - k}} \frac{1 - r^3}{r^2 + r^5}, \quad k_4 = \frac{5(r^3 - r^2 + r + 1)}{2r^3(1 + r^3)\theta^2(r + 1)}.$$

Hence, we obtain the following representation for C_1:

$$C_1 = \frac{\alpha_1 \cosh x + \alpha_2 \sinh x \cdot (\ln[(a - \tanh(x/2))/(a + \tanh(x/2))] + \alpha_3}{(a^2 - 1)\alpha_4(a^2 + 1 + (a^2 - 1)\cosh x)^2},$$

where

$$\alpha_1 = k_2 \frac{20a^2}{\theta^4}(1 + a^2) + k_3 \frac{30a^3}{\theta^5} + k_4 \frac{12a^2(1 + a^2)}{\theta^6},$$

$$\alpha_2 = 6k_4 a(a^2 - 1)^2/\theta^6, \qquad \alpha_3 = 4a^2(a^2 - 1)(5k_2 - 3k_4/\theta^2)/\theta^4, \qquad \alpha_4 = 15r(r + 1).$$

Plots of the functions C_0 and C_1 for $r = 1.2$ and $a = 1.0001$ are shown in Fig. 2.

4. Analysis of the Solution. We consider the flow regimes corresponding to the limiting values of the parameter k.

In the approximation obtained, the interface between the layers is given by the formula

$$y = 1 + \varepsilon a \frac{1 - \tanh^2(\varepsilon x/2)}{\theta(a^2 - \tanh^2(\varepsilon x/2))} + \varepsilon^2 C_1(x) + O(\varepsilon^3).$$

For fixed a or k, the wave amplitude is of the same order of magnitude as ε. In the limit $a \to 1$, as the point $(\mathrm{Fr}_1, \mathrm{Fr}_2)$ in the plane of Froude numbers approaches the straight line $\mathrm{Fr}_1 + \mathrm{Fr}_2 = 1$, the amplitude remains bounded. The crest of the solitary wave flattens, and its front near the point $x = a \ln[(a - 1)/(a + 1)]$ transforms to a bore (see Fig. 3). Here a_1 and a_2 are different values of the parameter a.

In the limit $a^{-1} \sim \varepsilon \to 0$, we obtain a Korteweg–de Vries wave:

$$y = 1 + \varepsilon^2 \theta^{-1} \cosh^{-2}(\varepsilon x) + o(\varepsilon^2).$$

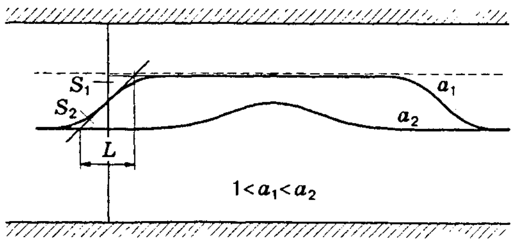

Fig. 3

It is possible to relate the parameters (ε, k) to the geometric parameters of the flow, e.g., to the amplitude A. As an additional parameter, it is convenient to use the width L of the wave front, which is defined as the difference of the abscissas of the points at which the inflectional tangent to the wave intersects the lines $y = 0$ and $y = 1/a$ (see Fig. 3), $0 < L \leqslant 4$. We note that if we draw a vertical line through the point of tangency, the areas of the curvilinear triangles S_1 and S_2 bounded by the plot of the function C_0 and by its asymptotes are equal. As a first approximation, we have

$$a = 2 - L/4, \qquad \varepsilon = A(2 - L/4)\sqrt{3(r+1)/(r^4 + r)}.$$

In terms of (A, L), the Froude numbers are given by

$$\mathrm{Fr}_1 = \frac{1}{r+1} + \frac{1}{32}\,\frac{80 - 16L + L^2}{r+1}\,A + O(A^3), \quad \mathrm{Fr}_2 = \frac{r}{r+1} - \frac{1}{32}\,\frac{80 - 16L + L^2}{r+1}\,A + O(A^3).$$

For $L = 4$, i.e., for the bore, we obtain

$$\mathrm{Fr}_1 = \frac{1}{r+1} + \frac{1}{r+1}A + O(A^3), \quad \mathrm{Fr}_2 = \frac{r}{r+1} - \frac{1}{r+1}A + O(A^3).$$

The effective length of the solitary wave that represents the distance between the points of inflection at the edges of the wave is given (in these variables) by

$$l = \frac{8 - L}{2}\ln\frac{12 - L}{4 - L}.$$

Conclusions. We proposed an algorithm for constructing a uniform asymptotic solution of the problem of internal waves in a two-layer fluid under a rigid lid. The approximation obtained can be used to prove the existence theorem for an exact solution, which substantiates the technique employed herein. The family of internal solitary waves considered here is of interest because for a fixed amplitude there are no restrictions on the effective wavelength. As a consequence, such waves can transfer as much energy as one likes.

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